Interface Stability in a Slowly Rotating Low-Gravity Tank

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The equilibrium configuration of a bubble in a rotating liquid confined by flat axial boundaries (baffles) is found. The maximum baffle spacing assuring bubble confinement is bounded from above by the natural length of a bubble in an infinite medium under the same conditions. Effects of nonzero contact angle are minimal. The problem of dynamic stability is posed. It can be solved in the limit of rapid rotation, for which the bubble is a long cylinder. Instability is to axisymmetric perturbations; nonaxisymmetric perturbations are stable. The stability criterion agrees with earlier results. (Extension to the general case is beyond the scope of this work.)

Nomenclature

a,b = integrals defined in Eq. (20)
c_1,c_2 = integration constants appearing in Eq. (4)
e = $\rho\Omega^2 R_M^3/8T$, a rotational Bond number
$\mathbf{e}_\phi$ = unit vector in the azimuthal direction
E = $\nu/\Omega L^2$, Ekman number
E_0 = energy constant in Eq. (3)
f = dimensionless deformation of equilibrium radius
g = function equal to radial component of the normal to the interface
h = L/R_m, ratio of length to maximum bubble diameter
H = Lagrangian defined in Eq. (3)
I = integral defined in Eq. (21)
k = dimensionless axial wave number
$\mathbf{k}$ = unit vector in the axial direction
$K(z)$ = generalized wave number in the axial direction
L = half-height (length) of the bubble
L^* = $L/V^{1/3}$, dimensionless half-height
m = azimuthal wave number
n = integer in quantized axial wave number
$\mathbf{n}$ = normal to the interface
n_r = radial component of normal to the interface
p = dimensionless perturbation pressure
P = pressure
r = radial coordinate
$R(z)$ = radial coordinate of equilibrium bubble
R_M = minimum bubble radius
R_m = maximum bubble radius
s = complex oscillation frequency
T = surface tension
$\mathbf{u}$ = dimensionless perturbation velocity
U = velocity
V = bubble volume
z = axial coordinate
β = R_m/R_M
β^* = $[\beta(\beta-\sin\theta)/(1-\beta\sin\theta)]^{1/2}$
Δ = $1/g$
θ = contact angle
ν = liquid kinematic viscosity
ρ = liquid density
Σ = $s+m$, apparent frequency in a rotating coordinate system
ϕ = azimuthal coordinate
Ω = rotation rate
Ω^* = $[\rho\Omega^2 V/8T]^{1/3}$, dimensionless rotation rate
$\langle\ \rangle$ = integral over bounding surface
$\langle\!\langle\ \rangle\!\rangle$ = integral over liquid-filled volume

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Introduction

ONE way to control large amounts of liquid in near Earth orbit (fuel for later burns, cryogens for experiments,...) is to rotate the container, holding the liquid against the outer walls. In many applications it is important not only to know where the vapor is, but to know that it is symmetric. A bubble small compared to the size of the container is free to move within the container, though remaining near the axis.

Earlier work[1,2] analyzed the behavior of a bubble confined by baffles perpendicular to the rotation axis. By integrating the ordinary differential equation governing the shape of a symmetric drop numerically, maximum bafffle spacings were estimated. This maximum length was taken to be the point at which the numerical integration scheme broke down, and it seemed advisable to put the conclusion on a firmer footing. Since Rosenthal's work[3] was discovered, it proved possible to extend his method to the case of a confined bubble. That extension is the main purpose of this paper.

A secondary purpose is to formulate the stability problem for the equilibrium configuration, and to show that unsteady interface perturbations, which we call inertial-capillary waves, can be unstable. This is of some general interest, because inertial waves (free oscillations of a rotating inviscid liquid) can be shown to be stable if the container in which they are confined is completely full. (See Greenspan's monograph,[4] for example.)

The limiting case of rapid rotation is accessible. For axisymmetric perturbations the conclusions are the same as those found by Rosenthal. Because the length is given we can construct a definite stability criterion. We also show that nonaxisymmetric perturbations are stable.

The general case is quite complex and beyond the scope of this paper.

Formulation

Let the liquid and its vapor be contained in a container rotating at Ω. Construct an r,ϕ,z cylindrical coordinate system with the z axis parallel to the rotation axis and ϕ positive in the direction of rotation. Denote the extent of the container in the direction parallel to Ω by $2L$. Suppose the end walls to be flat, and perpendicular to the rotation axis, so that they are located at $z = \pm L$. The container shape is otherwise arbitrary. Figure 1 is a sketch of the system.

Denote the density and kinematic viscosity of the liquid by ρ and ν. Let the interfacial tension be T. Imagine a basic state in which the liquid is in solid rotation and the vapor fills the center. The interface may intersect the walls symmetrically. If it does it will do so at a fixed contact angle, θ, assumed to be less than $\pi/2$. Let the flow in the liquid be a small departure from solid corotation, so that the velocity and pressure can be written

$$U = \Omega R_M (r e_\phi + u); \quad P = \rho \Omega^2 R_M^2 (r^2/2 + p) \tag{1}$$

Here the first stage in nondimensionalization is introduced; r and z are scaled by R_M, u by ΩR_M, and p by $\rho(\Omega R_M)^2$, where R_M denotes the maximum radius of the bubble, attained at its waist. Let R_m denote its minimum radius, and denote their ratio by $\beta(=R_m/R_M)$.

The choice of R_M as a length scale seems unwise from the point of view of results, since R_M is unknown. However it turns out to be the natural scaling for the problem. The numerical results will be presented in a more useful dimensionless scheme, introduced in the next section.

Rosenthal[3] studied the free (nonintersecting) bubble. He was able to obtain integral expressions for the volume and aspect ratio of the free bubble. The extension to the confined bubble situation is reasonably straightforward. One writes the length and volume as

$$L = \int_0^L \mathrm{d}z; \; V = 2\pi R_M^2 \int_0^L R^2(z')\mathrm{d}z' \tag{2}$$

where z' is a dimensional axial coordinate. These are converted to integrals of known functions over r by replacing $\mathrm{d}z$ by $(\mathrm{d}z/\mathrm{d}r)\mathrm{d}r$, where $\mathrm{d}z/\mathrm{d}r$ is found from the ordinary differential equation governing the interface. This can be obtained from a variational principle: minimize the difference between the rotational energy of the liquid-bubble system (confined in a finite volume containing the bubble) and its surface energy, subject to the constraint of constant fixed bubble volume.

The difference is just the usual Lagrangian of Hamilton's principle, with the surface energy playing the role of potential energy. It can be written

$$H = (1/2)(\pi\rho\Omega^2 R_M^5)\{E_0 - \int_0^h r^4 \mathrm{d}z\} - 4\pi T R_M^2 \int_0^h [1 - r'^2] r \mathrm{d}z \tag{3}$$

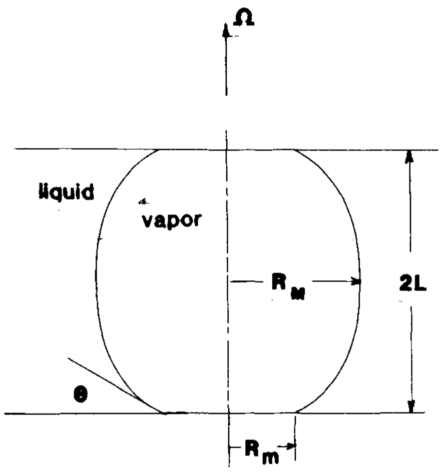

Fig. 1 Definition sketch of the constrained rotating bubble.

where E_0 is a constant depending on the volume of integration and $h = L/R_M$. The resulting differential equation for $r(z)$ is

$$[1 + r'^2]^{-(1/2)} = n_r = g(r,e,\beta) = c_1 r - e r^3 + c_2/r \tag{4}$$

where r' stands for $\mathrm{d}r/\mathrm{d}z$, $e = \rho\Omega^2 R_M^3/(8T)$, n_r is inserted to emphasize that g is the radial component of the outward normal to the bubble, and, after finding the two constants from the boundary conditions at $r = \beta$, 1, g can be conveniently rewritten as

$$g = [(1 - \beta\sin\theta)/r(1-\beta^2)] \times [r^* - \beta^{*2}] + e(1-r^2)(r-\beta^2)r \tag{5}$$

with

$$\beta^{*2} = \beta(\beta - \sin\theta)/(1 - \beta\sin\theta) \tag{6}$$

The constant c_1 is the Lagrange multiplier arising from the constraint of constant volume. The constant c_2 is the constant associated with the first integral of the variation. It vanishes for a free bubble. After conversion the two integrals may be written

$$L = R_M \int_\beta^1 g/[1-g^2]\mathrm{d}r, \qquad V = 2\pi R_M^3 \int_\beta^1 r^2 g/[1-g^2]\mathrm{d}r \tag{7}$$

In the free bubble ($\beta = 0$) case, these are expressible in terms of elliptic integrals of the first and second kind. In the general case they become hyperelliptic integrals. In both cases it seems simplest to integrate numerically. The integrands are well-behaved over the interval of integration, with a square root singularity at the upper limit.

Results and Discussion

One would like to be able to specify the physical variables ρ, Ω, T, V, and θ, and find L and β. Unfortunately this cannot be done directly, as the integrals give only an implicit relationship among the independent and dependent variables. What has been done is to calculate V and L as functions of e and β and invert the result to find R_M, from which the remaining dependent variables can be found.

The integrals were evaluated using $u = r^2$ as the variable of integration, extracting the square root singularity at $u = 1$ and integrating the rest using Bode's rule (a five point quartic spline scheme[5]) with 20 total nonoverlapping intervals (four splines). The absolute results are within one percent of the correct values, and the relative results are much more precise.

A summary of the results is shown in Fig. 2, which gives L^*, the length of the bubble normalized by the cube root of the bubble volume, as a function of Ω^*, the cube root of $\rho\Omega^2 V/(8T)(=[V/R_M^3]e)$. The contact angle has been taken to be zero. Calculations with nonzero contact angle show a negligible difference for $\theta < \pi/2$. (For $\theta > \pi/2$ a reformulation is necessary.)

The upper bold line is the $\beta = 0$ asymptote. It is identical to Rosenthal's result. There are no bubbles above this line. Baffle spacings above this line will not confine bubbles. The other curves on the figure are lines of constant e and constant β. It can be shown that

$$e < (1 + \beta^2 - 2\beta\sin\theta)/[2(1-\beta^2)^2] \tag{8}$$

is a necessary condition for the integral to be well-behaved. Thus 0.5 is on upper bound on e for $\beta = 0$, agreeing with the conclusion reached by Rosenthal. That asymptote is shown boldly on the figure.

Some Comments on Interface Stability

Imagine that the interface is disturbed, so that u and p introduced in Eq. (1) are nonzero. Denote the free surface per-

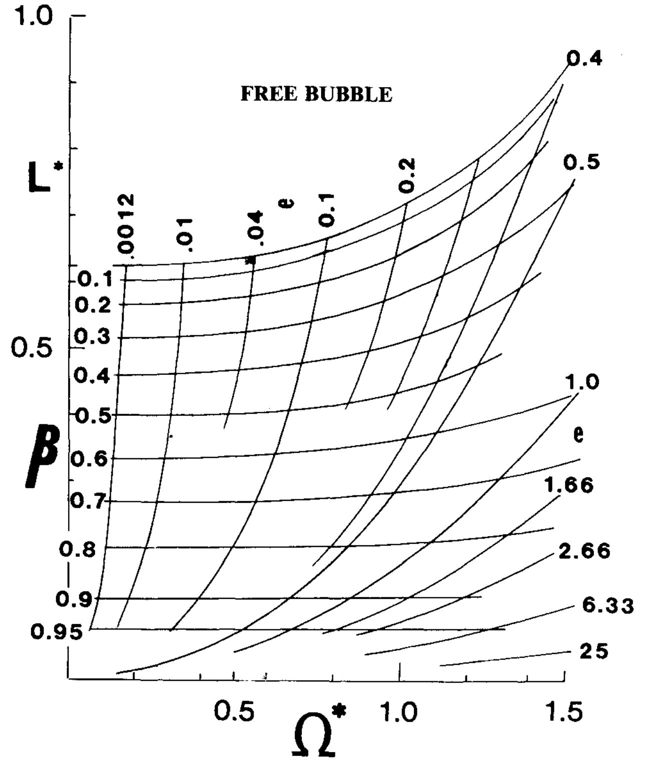

Fig. 2 Bubble shape map: e and β as functions of dimensionless length L^* and rotation rate Ω^*.

turbation by f, so that the dimensionless expression of the free surface becomes

$$r = R(z) + f(z,\phi,t) \tag{9}$$

where $R(z)$ denotes the solution to Eq. (4), the equilibrium nondimensional interface radius discussed above. (Note that the perturbation f cannot be independent of both z and ϕ for an incompressible liquid.) The perturbation must satisfy the Navier-Stokes equation. (It will be supposed that the Euler equations are adequate, as attention is to be focused on the possibility of inertial-surface tension instability.) The boundary conditions are that $\boldsymbol{n}\cdot\boldsymbol{u}$ vanish on the solid surfaces, and that the kinematic and pressure conditions be satisfied on the free surface.

The kinematic condition is

$$f_t - u + f_\phi + (v/r)f_\phi + wR'(z) = 0 \tag{10}$$

where u,v,w are the radial, azimuthal, and axial components of $\boldsymbol{u}$, subscripts are used to denote partial differentiation, and the prime denotes differentiation with respect to argument. The pressure condition is that the inner pressure exceed the outer pressure by T times the divergence of the unit normal (pointing from the vapor to the liquid). (In the general viscous case there is also a condition on the continuity of shear stresses.)

The unit normal is

$$n = \frac{[e_r - R'(z)e_z - \nabla f]}{\{1 + R'^2 + 2f_zR' + f_z^2 + [f_\phi/(R+f)]^2\}^{1/2}} \tag{11}$$

and the divergence operator becomes

$$\left(\frac{\partial}{\partial r} + \frac{1}{R+f}\right)e_r\cdot + \frac{1}{R+f}\frac{\partial}{\partial \phi}e_\phi\cdot + \frac{\partial}{\partial z}e_z\cdot \tag{12}$$

At "zeroth order," when the perturbation quantities p and $\boldsymbol{u}$ are zero, the solution is just that found above.

The perturbation flow satisfies the partial differential equation

$$u_t + u_{\phi_1} + u\cdot\nabla u + \nabla p = E\nabla^2 u$$
$$\nabla\cdot u = 0 \tag{13}$$

where $\boldsymbol{k}$ is a unit vector parallel to the rotation axis, $E = \nu/\Omega L^2$ is the usual Ekman number and the subsubscript 1 on the ϕ derivative means that one is to differentiate only the components of $\boldsymbol{u}$. (This term would appear as an apparent time derivative were the analysis done in a rotating coordinate system.) For the purposes of investigating the stability of this perturbation the equation will be linearized by dropping the advective term and viscosity will be neglected.

Complex notation can be introduced without loss of generality

$$u,p \rightarrow u(r),p(r)\exp[i(m\phi + st + K(z))] \tag{14}$$

where $K(z)$ reduces to kz in the limit of infinite rotation rate, when the boundary equation becomes separable in z as well as ϕ and t. The combination $s+m$ will be denoted by Σ.

The differential equations governing the perturbation in this linear inviscid limit are then

$$i\Sigma u + 2k\times u + \nabla p = 0$$
$$\nabla\cdot u = 0 \tag{15}$$

and the boundary conditions are that $\boldsymbol{n}\cdot\boldsymbol{u}$ vanish on the solid boundaries, and that

$$n\cdot u = -i\Sigma f$$
$$p = -Rf - \{(R'/R\Delta^3)f_z + (f_z/\Delta^3)_z + f_{\phi\phi}/(R^2\Delta) - f/(R^2\Delta)\} \tag{16}$$

where $\Delta^2 = 1 + R'^2 = 1/g^2$ on the free surface.

Multiply the momentum equation by the complex conjugate of $\boldsymbol{u}$ and integrate over the liquid volume. A simple integration by parts in the third term leads to

$$i\Sigma \ll u\cdot u^*\gg + 2\ll k\cdot(u\times u^*)\gg - \langle n\cdot u^* p\rangle = 0 \tag{17}$$

where the asterisk denotes complex conjugate, $\ll\ldots\gg$ denotes the volume integral, and $\langle\ldots\rangle$ denotes the integral over the interface. The minus sign appears because $\boldsymbol{n}$ points into the liquid. Manipulation, relying on integration by parts, leads to a pair of equations for the real and imaginary parts of Σ,Σ_R and Σ_I:

$$\Sigma_R\{\ll u\cdot u^*\gg + e^{-1}[aa^* - bb^* + \mathrm{Re}(I)]\} + \Sigma_I \mathrm{Im}(I)/e = 2\Omega i \ll k\cdot(u\times u^*)\gg \tag{18}$$

$$\Sigma_R \mathrm{Im}(I)/e + \Sigma_I\{\ll u\cdot u^*\gg - e^{-1}[aa^* - bb^* + \mathrm{Re}(I)]\} = 0 \tag{19}$$

where the right-hand side of the first equation is real and the constants aa^*, bb^* are given by

$$aa^* = 2\pi\int_0^h [ff^*/(R\Delta^2)]\mathrm{d}z$$

$$bb^* = 2\pi\int_0^h [(R/\Delta^4)f_z f_z^* + (f_\phi f_\phi^* + \mathrm{eff}^*)/R\Delta^2]\mathrm{d}z \tag{20}$$

The integral I is

$$I = 2\pi\int[RR'R''f^*f_z)/\Delta^6]\mathrm{d}z \tag{21}$$

Making use of the complex notation one can show that the real and imaginary parts of I are

$$\mathrm{Re}(I) = \pi\int_0^h [RR'R''''(ff^*)_z/\Delta^6]\mathrm{d}z$$

$$\mathrm{Im}(I) = 2\pi\int_0^h [RR'R''ff^*K'(z)/\Delta^6]\mathrm{d}z \tag{22}$$

In the simple case of rapid rotation the problem is asymptotically separable in z, and $K(z)$ can be replaced by kz. In this case $R\rightarrow$constant, the equations uncouple, and instability is possible if the coefficient of $Im(\Sigma)$ in Eq. (19) vanishes.

The coefficient cannot vanish for nonzero m; nonaxisymmetric disturbances are stable. This appears to be a new result, even for the unconfined bubble. If $m=0$, the flow will be stable for $k^2>1$. Finally, for $k^2<1$, the flow will be stable unless $e<1-k^2$. This is exactly the conclusion reached by Rosenthal. The endwall boundary conditions fix $k(=n\pi/h)$.

In this limit $V=2\pi R_M^2 L$, so that $h^2=2\pi L^3/V$, $k^2=(n\pi)^2 V \div (2L^3)$ and after some manipulation the asymptotic necessary condition for instability can be written

$$L^* = \Omega^{*3}/(8\pi)^{(1/2)}\{1+[1+4\pi^2/\Omega^{*6}]^{(1/2)}\} \tag{23}$$

The reader should remember that this applies only in the limit that $\Omega^*\rightarrow\infty$. It is tempting to suppose that the general case is qualitatively similar, but a great deal of work is necessary to prove that.

Summary and Conclusions

We have shown that the smallest bubble which can be confined between two "horizontal" baffles is the equilibrium Rosenthal bubble of length $2L$. The maximum and minimum radii of larger bubbles can be found from Fig. 2. The dimensionless numbers L^* and Ω^* can be found from the independent variables. These can be used to enter the diagram, finding e, which gives R_M, and β, which gives R_m.

We have formulated the stability problem for perturbations to the equilibrium bubble. Specific conclusions are possible for rapid rotation. In that case, we conclude that nonaxisymmetric perturbations are stable and that short wavelength axial perturbations are also stable.

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